

# Off-diagonal Gluon Mass Generation and Infrared Abelian Dominance in Maximally Abelian Gauge in SU(3) Lattice QCD

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## Abstract

In SU(3) lattice QCD formalism, we propose a method to extract gauge fields from link-variables analytically. With this method, we first study effective mass generation of off-diagonal gluons and infrared Abelian dominance in the maximally Abelian (MA) gauge in the SU(3) case. Using SU(3) lattice QCD, we investigate the propagator and the effective mass of the gluon fields in the MA gauge with  $U(1)_3 \times U(1)_8$  Landau gauge fixing. The Monte Carlo simulation is performed on  $16^4$  at  $\beta = 5.7, 5.8$  and  $6.0$  at the quenched level. The off-diagonal gluons behave as massive vector bosons with the approximate effective mass  $M_{\text{off}} \simeq 1.0 - 1.2 \text{ GeV}$  in the region of  $r = 0.2 - 0.8 \text{ fm}$ , and the propagation is limited within a short range, while the propagation of diagonal gluons remains even in a large range. In this way, infrared Abelian dominance is shown in terms of short-range propagation of off-diagonal gluons. Furthermore, we investigate the functional form of the off-diagonal gluon propagator. The functional form is well described by the four-dimensional Euclidean Yukawa-type function  $e^{-m_{\text{off}} r}/r$  with  $m_{\text{off}} \simeq 1.5 - 1.6 \text{ GeV}$  for  $r = 0.1 - 0.8 \text{ fm}$ . This also indicates that the spectral function of off-diagonal gluons has the negative-value region.

*Keywords:*

## 1. Introduction

Quantum chromodynamics (QCD) is the fundamental gauge theory of the strong interaction based on quarks and gluons. There are a variety of nonperturbative phenomena in low energy QCD such as color confinement and chiral symmetry breaking. These nonperturbative phenomena have been studied both in analytical frameworks and in lattice QCD [1–3].

On the quark-confinement mechanism, Nambu, 't Hooft and Mandelstam suggested the dual-superconductor picture [4]. This picture is based on the electromagnetic duality and the analogy with the one-dimensional squeezing of the magnetic flux in the type-II superconductor. In this picture, there occurs color magnetic monopole condensation, and then the color-electric flux between the quark and the antiquark is squeezed as a one-dimensional tube due to the dual Higgs mechanism. From the viewpoint of the dual-superconductor picture in QCD, however, there are two assumptions of Abelian dominance [5, 6] and monopole condensation. Here, Abelian dominance means that only the diagonal gluon component plays the dominant role for the nonperturbative QCD phenomena like confinement.

The maximally abelian (MA) gauge has mainly been investigated from the viewpoint of the dual-superconductor picture [7–17] and the various lattice QCD Monte Carlo simulations show that the MA gauge fixing seems to support these assumptions [9–17].

According to these studies, the diagonal gluons seem to be significant to the infrared QCD physics, which is called “infrared Abelian dominance”. Infrared Abelian dominance means that off-diagonal gluons do not contribute to infrared QCD. Therefore, the essence of infrared Abelian dominance is the behavior of the off-diagonal gluon propagator.

The gluon propagators in the MA gauge has been investigated in SU(2) lattice Monte Carlo simulations [16, 17]. To investigate the gluon propagators in the MA gauge, the gluons would be extracted exactly from the link-variables, because the link-variable cannot be expanded for a small lattice spacing due to the large fluctuation of gluons. In SU(2) lattice case, the extraction is easy to be done without any approximation, because of the SU(2) property. With this extraction, the SU(2) lattice simulation suggests that the off-diagonal gluons do not propagate in the infrared region due to the effective mass  $M_{\text{off}} \simeq 1.2 \text{ GeV}$ , while the diagonal gluon widely propagates [16].

The aim of this paper is to propose a method to extract the gluons from the link-variable directly and generally in SU(3) lattice QCD, and to investigate the gluon propagators in the MA gauge.

## 2. Formalism to extract gluon fields from link-variables

In the Landau gauge, the link-variables can be expanded for the small lattice spacing within the scaling region and the gluons are easily obtained from link-variables approximately [18–21]. In Euclidean QCD, the Landau gauge is defined by mini-

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mizing the global quantity,

$$R \equiv \int d^4x A_\mu^a(x) A_\mu^a(x), \quad (1)$$

by the gauge transformation, where  $A_\mu^a(x) \in \mathbf{R}$  ( $a = 1 \cdots 8$ ) are gluon fields. This means that the gauge fluctuation is maximally suppressed in the Landau gauge. On the lattice, the gauge fields are expressed by the link-variables  $U_\mu(x) \equiv e^{iagA_\mu(x)} \in \text{SU}(3)$  with the lattice spacing  $a$  and the gauge coupling constant  $g$ . Therefore, in the Landau gauge,  $|agA_\mu(x)| \ll 1$  would be satisfied within the scaling region and the gluon fields would be extracted from the link-variables,

$$A_\mu(x) \simeq \frac{1}{2iag} [U_\mu(x) - U_\mu^\dagger(x)] - (\text{trace part}). \quad (2)$$

In other gauge, however, it is not straightforward to extract the gluons from the link-variables.

In this section, we consider a useful and general method to extract the gauge fields analytically and exactly from the link-variables whether  $|agA_\mu(x)| \ll 1$  is satisfied or not [19].

To this end, we first define the hermite matrix,

$$\begin{aligned} \Lambda &\equiv \frac{1}{2i} (U - U^\dagger) \\ &= \frac{1}{2i} (e^{iagA} - e^{-iagA}) \equiv \sin agA. \end{aligned} \quad (3)$$

For simplicity, we have omitted the Lorentz index and space-time arguments.

Arbitrary hermite matrix  $\Lambda$  can be diagonalized by a unitary transformation,

$$\Lambda_d \equiv \Omega \Lambda \Omega^\dagger = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}, \quad (4)$$

where  $\Omega \in \text{SU}(3)$ . We can obtain the eigenvalues  $\lambda_i$  ( $i = 1, 2, 3$ ) by solving

$$\det(x1 - \Lambda) = 0. \quad (5)$$

This is a cubic equation on  $x$ . The eigenvalues  $\lambda_i$ , i.e., The solutions of the equation are

$$x_{0,\pm} = z_{0,\pm} \sqrt{\alpha^2 + \beta/3} + \alpha, \quad (6)$$

where

$$\begin{aligned} z_0 &\equiv e^{i\theta/3} + e^{-i\theta/3} \\ z_\pm &\equiv e^{i(\theta \pm 2\pi)/3} + e^{-i(\theta \pm 2\pi)/3}. \end{aligned} \quad (7)$$

The derivation of the solution and the notation for Eqs. (6) and (7) are given in Appendix A. Thus,  $\lambda_i$  is obtained.

The unitary matrix  $\Omega$  can be also derived as follows. By solving  $M\vec{e}_i = \lambda_i\vec{e}_i$ , we obtain eigenvectors  $\vec{e}_i = {}^t(x_i, y_i, z_i)$  ( $i = 1, 2, 3$ ),  $|\vec{e}| = 1$ . We assume that  $z_i$  is nonzero without loss of generality and rescale it by  $1/z_i$ ,

$$\Lambda \begin{pmatrix} x_i/z_i \\ y_i/z_i \\ 1 \end{pmatrix} = \lambda_i \begin{pmatrix} x_i/z_i \\ y_i/z_i \\ 1 \end{pmatrix}. \quad (8)$$

This is solved easily as

$$\begin{pmatrix} x_i/z_i \\ y_i/z_i \end{pmatrix} = -\{(\Lambda_{11} - \lambda_i)(\Lambda_{22} - \lambda_i) - \Lambda_{12}\Lambda_{21}\}^{-1} \cdot \begin{pmatrix} \Lambda_{22} - \lambda_i & -\Lambda_{12} \\ -\Lambda_{21} & \Lambda_{11} - \lambda_i \end{pmatrix} \begin{pmatrix} \Lambda_{13} \\ \Lambda_{23} \end{pmatrix}. \quad (9)$$

From the normalization condition  $|\vec{e}_i| = 1$ , we obtain  $\Omega^\dagger = (\vec{e}_1^\dagger, \vec{e}_2^\dagger, \vec{e}_3^\dagger)$ .

When we diagonalize  $M$  with the unitary matrix  $\Omega$ ,  $A$  is also diagonalized,

$$\begin{aligned} \Lambda_d &= \Omega \Lambda \Omega^\dagger = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \\ &= \sin(ag\Omega \Lambda \Omega^\dagger) \equiv \begin{pmatrix} \sin \theta_1 & & 0 \\ & \sin \theta_2 & \\ 0 & & \sin \theta_3 \end{pmatrix}, \end{aligned} \quad (10)$$

where  $-\pi/2 \leq \theta_i \leq \pi/2$  ( $i = 1, 2, 3$ ) is taken. Therefore we can derive gluon fields  $A$  from link-variables  $U$  analytically,

$$\begin{aligned} \Omega \Lambda \Omega^\dagger &= \frac{1}{ag} \begin{pmatrix} \theta_1 & & 0 \\ & \theta_2 & \\ 0 & & \theta_3 \end{pmatrix} \\ \Rightarrow A &= \frac{1}{ag} \Omega^\dagger \begin{pmatrix} \theta_1 & & 0 \\ & \theta_2 & \\ 0 & & \theta_3 \end{pmatrix} \Omega. \end{aligned} \quad (11)$$

This formalism is quite general, because the derivation is correct with any gauge and even without any gauge fixing.

### 3. SU(3) lattice QCD results for gluon propagators in the MA gauge

Using the SU(3) lattice QCD, we calculate the gluon propagators in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing. In the MA gauge, to investigate the gluon propagators, we use the gluon fields extracted directly from the link-variables. The Monte Carlo simulation is performed on the  $16^4$  lattice with  $\beta = 5.7, 5.8$  and  $6.0$  at the quenched level. All measurements are done every 500 sweeps after a thermalization of 10,000 sweeps using the pseudo heat-bath algorithm. We prepare 50 gauge configurations for the calculation at each  $\beta$ . Error is estimated with the jackknife analysis. The MA gauge fixing is performed by the maximization of

$$R_{\text{MA}} \equiv \sum_s \sum_{\mu=1}^4 \text{tr} [U_\mu(x) \vec{H} U_\mu^\dagger(x) \vec{H}], \quad (12)$$

where  $\vec{H} = (T_3, T_8)$  is the Cartan generator. In this gauge fixing, there remains  $U(1)_3 \times U(1)_8$  gauge symmetry. After the Cartan decomposition for the SU(3) link-variables as  $U_\mu(x) \equiv M_\mu(x) u_\mu(x)$  with  $u_\mu(x) \equiv e^{i(\theta^3(x)T^3 + \theta^8(x)T^8)} \in U(1)_3 \times U(1)_8$  and  $M_\mu(x) = e^{i \sum_{a \neq 3,8} \theta^a(x) T^a} \in \text{SU}(3)/U(1)_3 \times U(1)_8$ , the residual gauge fixing is performed by the maximization of

$$R_{\text{U(1)L}} \equiv \sum_s \sum_{\mu=1}^4 \text{Re tr} [u_\mu(x)]. \quad (13)$$

At  $\beta = 5.7, 5.8$  and  $6.0$ , the lattice spacings  $a$  are estimated as  $a \simeq 0.186\text{fm}, 0.152\text{fm}$  and  $0.104\text{fm}$ , respectively, which lead to the string tension  $\sigma \simeq 0.89\text{GeV/fm}$  in the inter-quark potential [21].

Here, we study the Euclidean scalar-type propagator of the diagonal (Abelian) gluon as

$$G_{\mu\mu}^{\text{Abel}}(r) \equiv \frac{1}{2} \sum_{a=3,8} \langle A_\mu^a(x) A_\mu^a(y) \rangle, \quad (14)$$

and that of the off-diagonal gluon as

$$G_{\mu\mu}^{\text{off}}(r) \equiv \frac{1}{6} \sum_{a \neq 3,8} \langle A_\mu^a(x) A_\mu^a(y) \rangle. \quad (15)$$

These scalar-type propagators are expressed as the function of the four-dimensional Euclidean distance  $r \equiv \sqrt{(x_\mu - y_\mu)^2}$ .

We show in Fig. 1 the lattice QCD result for the diagonal gluon propagator  $G_{\mu\mu}^{\text{Abel}}(r)$  and the off-diagonal gluon propagator  $G_{\mu\mu}^{\text{off}}(r)$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing. In the MA gauge,  $G_{\mu\mu}^{\text{Abel}}(r)$  and  $G_{\mu\mu}^{\text{off}}(r)$  manifestly differ. The diagonal-gluon propagator  $G_{\mu\mu}^{\text{Abel}}(r)$  takes a large value even at the long distance. In fact, the diagonal gluons  $A_\mu^3, A_\mu^8$  in the MA gauge propagate over the long distance. On the other hand, the off-diagonal gluon propagator  $G_{\mu\mu}^{\text{off}}(r)$  rapidly decreases and is negligible for  $r \gtrsim 0.4\text{ fm}$  in comparison with  $G_{\mu\mu}^{\text{Abel}}(r)$ . Then, the off-diagonal gluons  $A_\mu^a$  ( $a \neq 3, 8$ ) seem to propagate only within the short range as  $r \lesssim 0.4\text{ fm}$ . Thus, “infrared abelian dominance” is found in the MA gauge.

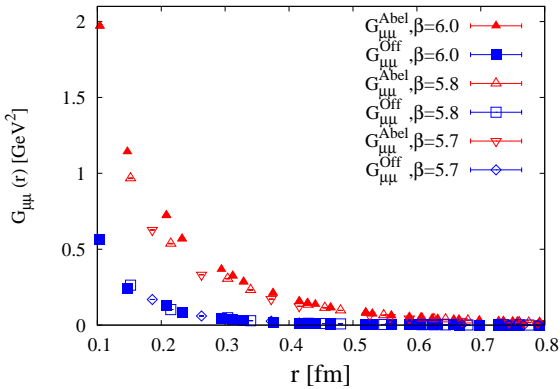


Figure 1: The SU(3) lattice QCD results of the scalar-type gluon propagators  $G_{\mu\mu}^{\text{Abel}}(r)$  and  $G_{\mu\mu}^{\text{off}}(r)$  as the function of  $r \equiv \sqrt{(x_\mu - y_\mu)^2}$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing in the physical unit. The Monte Carlo simulation is performed on the  $16^4$  lattice with  $\beta = 5.7, 5.8$  and  $6.0$ . The diagonal-gluon propagator  $G_{\mu\mu}^{\text{Abel}}(r)$  takes a large value even at the long distance. On the other hand, the off-diagonal gluon propagator  $G_{\mu\mu}^{\text{off}}(r)$  rapidly decreases.

#### 4. Estimation of the off-diagonal gluon mass in the MA gauge

Next, we investigate the effective gluon mass. We start from the Lagrangian of the free massive vector field  $A_\mu$  with the mass

$M \neq 0$  in the Proca formalism,

$$\mathcal{L} = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}M^2 A_\mu A_\mu, \quad (16)$$

in the Euclidean metric. The scalar-type propagator  $G_{\mu\mu}(r; M)$  can be expressed with the modified Bessel function  $K_1(z)$  as

$$\begin{aligned} G_{\mu\mu}(r; M) &= \langle A_\mu(x) A_\mu(y) \rangle \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot (x-y)} \frac{1}{k^2 + M^2} \left( 4 + \frac{k^2}{M^2} \right) \\ &= 3 \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot (x-y)} \frac{1}{k^2 + M^2} + \frac{1}{M^2} \delta^4(x-y) \\ &= \frac{3}{4\pi^2} \frac{M}{r} K_1(Mr) + \frac{1}{M^2} \delta^4(x-y). \end{aligned} \quad (17)$$

In the infrared region with large  $Mr$ , Eq. (17) reduces to

$$G_{\mu\mu}(r; M) \simeq \frac{3\sqrt{M}}{2(2\pi)^{\frac{3}{2}}} \frac{e^{-Mr}}{r^{\frac{3}{2}}}, \quad (18)$$

using the asymptotic expansion,

$$K_1(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3}{2} + n)}{n! \Gamma(\frac{3}{2} - n)} \frac{1}{(2z)^n}, \quad (19)$$

for large  $\text{Re } z$ .

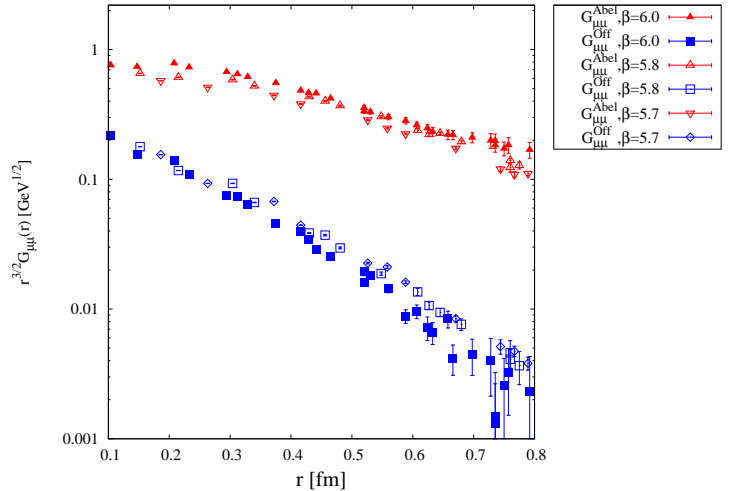


Figure 2: The logarithmic plot of  $r^{3/2} G_{\mu\mu}^{\text{off}}(r)$  and  $r^{3/2} G_{\mu\mu}^{\text{Abel}}(r)$  as the function of the distance  $r$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing, using the SU(3) lattice QCD with  $16^4$  at  $\beta = 5.7, 5.8$  and  $6.0$ .

In Fig. 2, we show the logarithmic plot of  $r^{3/2} G_{\mu\mu}^{\text{off}}(r)$  and  $r^{3/2} G_{\mu\mu}^{\text{Abel}}(r)$  as the function of the distance  $r$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing. From the linear slope on  $r^{3/2} G_{\mu\mu}^{\text{off}}(r)$ , the effective off-diagonal gluon mass  $M_{\text{off}}$  is estimated. We summarize in Table 1 the effective off-diagonal gluon mass  $M_{\text{eff}}$  obtained from the slope analysis at  $\beta = 5.7, 5.8$  and  $6.0$ . Therefore, the off-diagonal gluons seem to have a large mass  $M_{\text{off}} \simeq 1.0 - 1.2\text{GeV}$ . This result approximately coincides with SU(2) lattice calculation [16].

Table 1: Summary table of conditions and results in SU(3) lattice QCD. In the MA gauge, the off-diagonal gluons seem to have a large effective mass  $M_{\text{off}} \approx 1.0 - 1.2\text{GeV}$  and the functional form in the range of  $r = 0.1 - 0.8\text{fm}$  is well described with the four-dimensional Euclidean Yukawa function  $\sim \exp(-m_{\text{off}}r)/r$  with  $m_{\text{off}} \approx 1.5 - 1.6\text{GeV}$ .

lattice size	$\beta$	$a[\text{fm}]$	$M_{\text{off}}[\text{GeV}]$	$m_{\text{off}}[\text{GeV}]$
$16^4$	5.7	0.186	1.0	1.5
	5.8	0.152	1.0	1.5
	6.0	0.104	1.2	1.6

Finally in this section, we discuss the relation between infrared abelian dominance and the off-diagonal gluon mass. Due to the large effective mass  $M_{\text{off}}$ , the off-diagonal gluon propagation is restricted within about  $M_{\text{off}}^{-1} \approx 0.2\text{ fm}$  in the MA gauge. Therefore, at the infrared scale as  $r \gg 0.2\text{ fm}$ , the off-diagonal gluons  $A_\mu^a$  ( $a \neq 3, 8$ ) cannot mediate the long-range force, and only the diagonal gluons  $A_\mu^3, A_\mu^8$  can mediate the long-range interaction in the MA gauge. In fact, in the MA gauge, the off-diagonal gluons are expected to be inactive due to the large mass  $M_{\text{off}}$  in the infrared region in comparison with the diagonal gluons. Then, infrared abelian dominance holds within  $r \gg M_{\text{off}}^{-1}$ .

## 5. The analysis of the functional form of the off-diagonal gluon propagator in the MA gauge

In this section, we investigate the functional form of the off-diagonal gluon propagator in the MA gauge in SU(3) lattice QCD. In the previous section, we compare the gluon propagator with the massive vector boson propagator and estimate the gluon mass. In fact, the gluon propagator would not be described by a simple massive propagator Eq. (17) in whole region of  $r = 0.1 - 0.8\text{fm}$ .

There is the similar situation in the Landau gauge [21]. The functional form of the gluon propagator cannot be described by  $\exp(-Mr)/r^{3/2}$  with an effective mass  $M$  in whole region of  $r = 0.1 - 1.0\text{fm}$ . The appropriate form is the four-dimensional Euclidean Yukawa-type function  $\exp(-mr)/r$  with a mass parameter  $m$ .

In the same way, in the MA gauge, we also compare the gluon propagator with the four-dimensional Euclidean Yukawa function. In Fig. 3, we show the logarithmic plot of  $rG_{\mu\mu}^{\text{off}}(r)$  and  $rG_{\mu\mu}^{\text{Abel}}(r)$  as the function of the distance  $r$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing. Note that the logarithmic plot of  $rG_{\mu\mu}^{\text{off}}(r)$  are almost linear in the region of  $r = 0.1 - 0.8\text{fm}$  and then the off-diagonal gluon propagator is well expressed by the four-dimensional Euclidean Yukawa function in this region,

$$G_{\mu\mu}^{\text{off}}(r) \simeq A \frac{e^{-m_{\text{off}}r}}{r}, \quad (20)$$

with a mass parameter  $m_{\text{off}}$  and a dimensionless constant  $A$ . The best-fit mass parameter  $m_{\text{off}}$  is given in Table 1 at each  $\beta = 5.7, 5.8$  and  $6.0$ .

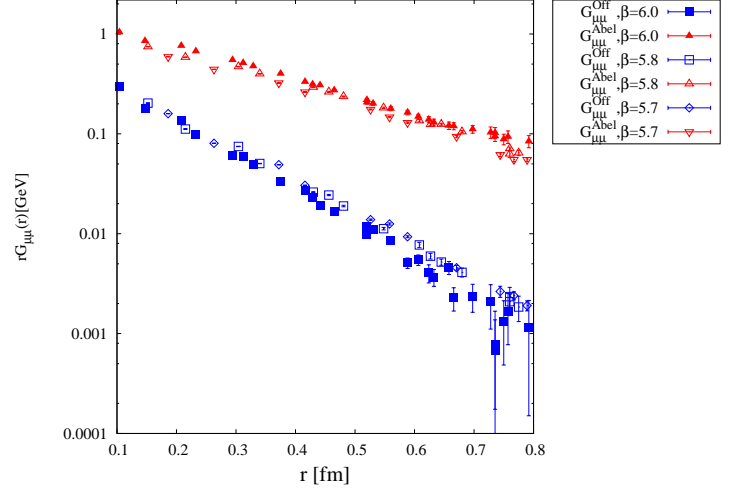


Figure 3: The logarithmic plot of  $rG_{\mu\mu}^{\text{off}}(r)$  and  $rG_{\mu\mu}^{\text{Abel}}(r)$  as the function of the distance  $r$  in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing, using the SU(3) lattice QCD with  $16^4$  at  $\beta = 5.7, 5.8$  and  $6.0$ . For  $rG_{\mu\mu}^{\text{off}}(r)$ , the approximate linear correspondence is found.

We comment on the four-dimensional Euclidean Yukawa-type propagator [21]. If the functional form of the off-diagonal gluon is well described by the four-dimensional Yukawa function, we analytically calculate the off-diagonal zero-spatial-momentum propagator,

$$D_0^{\text{off}}(t) \equiv \int d^3x G_{\mu\mu}^{\text{off}}(r), \quad (21)$$

and obtain the spectral function  $\rho(\omega)$  by the inverse Laplace transformation. Also in the MA gauge, the spectral function is found to have the negative-value region as in the Landau gauge [18, 20, 21],

$$\begin{aligned} \rho(\omega) = & - \frac{4\pi A m_{\text{off}}}{(\omega^2 - m_{\text{off}}^2)^{3/2}} \theta(\omega - m_{\text{off}}) \\ & + \frac{4\pi A / \sqrt{2m_{\text{off}}}}{(\omega - m_{\text{off}})^{1/2}} \delta(\omega - m_{\text{off}}). \end{aligned} \quad (22)$$

## 6. Summary and Concluding Remarks

We have first studied the gluon propagators in the MA gauge with the  $U(1)_3 \times U(1)_8$  Landau gauge fixing using the SU(3) lattice QCD. To investigate the gluon propagators in the MA gauge, we have considered to derive the gluon fields from the SU(3) link-variables. In this method, the gauge fields have been extracted by diagonalizing the link-variables and taking the logarithm. Owing to this method, any quantity expressed by the gluon fields can be calculated directly from link-variables, even if  $|agA_\mu(x)| \ll 1$  does not satisfy. As one of the general merits of this method, we can directly check the correspondence between gluon fields and the continuum gauge fixing in arbitrary lattice gauge fixing performed with link-variables. In principle, with this method, continuum gauge fixing with gluon fields can be also performed directly.

With this method, we have measured the Euclidean scalar-type propagators  $G_{\mu\mu}(r)$  of the diagonal and the off-diagonal gluons, and found the infrared Abelian dominance. The Monte Carlo simulation is performed on the  $16^4$  lattice with  $\beta = 5.7, 5.8$  and  $6.0$  at the quenched level. We have found that the off-diagonal gluons behave as massive vector bosons with the effective mass  $M_{\text{off}} \simeq 1.0 - 1.2$  GeV for  $r = 0.2 - 0.8$  fm. The effective gluon mass has been estimated from the linear fitting analysis of the logarithmic plot of  $r^{3/2}G_{\mu\mu}^{\text{off}}(r)$ . Due to the large value, the finite-size effect for the off-diagonal gluon mass is expected to be ignored. The large gluon mass shows that the off-diagonal gluons cannot mediate the interaction over the large distance as  $r \gg M_{\text{off}}^{-1}$ , and such an infrared inactivity of the off-diagonal gluons would lead infrared Abelian dominance in the MA gauge.

On the other hand, from the behavior of the diagonal gluon propagator  $G_{\mu\mu}^{\text{Abel}}(r)$  and  $r^{3/2}G_{\mu\mu}^{\text{Abel}}(r)$ , the diagonal gluons seem to behave as light or massless particles. However, for the detailed argument on  $G_{\mu\mu}^{\text{Abel}}(r)$ , one should consider the finite size effect more carefully, because the diagonal gluons would propagate over the long distance beyond the lattice size.

Finally, we have also investigated the functional form of the scalar-type propagator in the MA gauge. We show that the off-diagonal gluon propagator is well described by the four-dimensional Euclidean Yukawa-type form with the mass parameter  $m_{\text{off}} \simeq 1.5 - 1.6$  GeV in the region of  $r = 0.1 - 0.8$  fm. Then, this indicates that the spectral function  $\rho(\omega)$  of the off-diagonal gluons in the MA gauge has the negative-value region as in the Landau gauge.

On the other hand, the functional form of the diagonal gluon propagator seems to be the four-dimensional Euclidean Yukawa function with the lighter mass parameter. However, to discuss the functional form clearly, the finite size effect is to be checked carefully just like the estimation of the diagonal effective gluon mass.

In this study, we investigate the scalar-type off-diagonal gluon propagator. To be strict, the off-diagonal gluon propagator consists of two scalar functions corresponding to longitudinal and transverse components. Therefore, we will investigate each effective mass and the functional form of these components.

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## Appendix A. The eigenvalues of 3×3 hermite matrix

In this Appendix we derive the solution for the cubic equation,

$$\det(x1 - \Lambda) = 0, \quad (\text{A.1})$$

where  $\Lambda$  is an hermite matrix.

$$\begin{aligned} \det(x1 - \Lambda) &= \begin{vmatrix} x - \Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\ -\Lambda_{21} & x - \Lambda_{22} & -\Lambda_{23} \\ -\Lambda_{31} & -\Lambda_{32} & x - \Lambda_{33} \end{vmatrix} \\ &= x^3 - x^2(\Lambda_{11} + \Lambda_{22} + \Lambda_{33}) \\ &\quad + x(\Lambda_{22}\Lambda_{33} + \Lambda_{33}\Lambda_{11} + \Lambda_{11}\Lambda_{22} \\ &\quad - |\Lambda_{23}|^2 - |\Lambda_{31}|^2 - |\Lambda_{12}|^2) \\ &\quad - \det\Lambda = 0 \\ &\Rightarrow x^3 - 3\alpha x^2 - \beta x - \gamma = 0, \end{aligned} \quad (\text{A.2})$$

where  $3\alpha = \text{Tr}\Lambda \in \mathbf{R}$ ,  $\beta = -(\Lambda_{22}\Lambda_{33} + \Lambda_{33}\Lambda_{11} + \Lambda_{11}\Lambda_{22} - |\Lambda_{23}|^2 - |\Lambda_{31}|^2 - |\Lambda_{12}|^2) \in \mathbf{R}$ ,  $\gamma = \det\Lambda \in \mathbf{R}$ . Here, when we define  $x = y + \alpha$ , this equation is rewritten,

$$y^3 - (3\alpha^2 + \beta)y - (2\alpha^3 + \alpha\beta + \gamma) = 0, \quad (\text{A.3})$$

where

$$3\alpha^2 + \beta = \frac{1}{6}\{(\Lambda_{11} - \Lambda_{22})^2 + (\Lambda_{22} - \Lambda_{33})^2 + (\Lambda_{33} - \Lambda_{11})^2 + 6(|\Lambda_{23}|^2 + |\Lambda_{31}|^2 + |\Lambda_{12}|^2)\}.$$

For simplicity, we define  $p \equiv \sqrt{\alpha^2 + \beta/3} \in \mathbf{R}$ ,  $q \equiv 2\alpha^3 + \alpha\beta + \gamma \in \mathbf{R}$ ,

$$y^3 - 3p^2y - q = 0. \quad (\text{A.4})$$

If  $p \neq 0$ , we can rescale it by  $1/p$ . When we define  $y = pz$ , this equation is rewritten,

$$z^3 - 3z - 2r = 0, \quad (\text{A.5})$$

where  $2r \equiv qp^{-3} \in \mathbf{R}$ . When  $z = u + v$ ,

$$u^3 + v^3 + 3uv(u + v) - 3(u + v) - 2r = 0 \quad (\text{A.6})$$

$$\Leftrightarrow u^3 + v^3 - 2r + 3(u + v)(uv - 1) = 0. \quad (\text{A.7})$$

Without loss of generality, we can determine  $uv = 1$  and obtain

$$u^3 + v^3 = 2r \quad (\text{A.8})$$

$$u^3 v^3 = 1. \quad (\text{A.9})$$

These solution are obtained by solving

$$t^2 - 2rt + 1 = 0. \quad (\text{A.10})$$

Therefore this solution is

$$\begin{aligned} t &= r \pm i\sqrt{1 - r^2} \\ &\equiv e^{\pm i\theta}, \end{aligned} \quad (\text{A.11})$$

where  $r^2 \leq 1$  could be proven. By using this,  $z$  can be obtained. Note that the solutions of  $z$  are real, because the eigenvalues of  $\Lambda$  are real.

$$z_0 \equiv e^{i\theta/3} + e^{-i\theta/3} \quad (\text{A.12})$$

$$z_{\pm} \equiv e^{i(\theta \pm 2\pi)/3} + e^{-i(\theta \pm 2\pi)/3}. \quad (\text{A.13})$$

Therefore we can obtain the solutions of Eq. (A.1),

$$\begin{aligned} x_{0,\pm} &= pz_{0,\pm} + \alpha \\ &= z_{0,\pm} \sqrt{\alpha^2 + \beta/3} + \alpha. \end{aligned} \quad (\text{A.14})$$

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